Improving the Performance of Linear Inverse Solutions by Inverting the Resolution Matrix

R. Grave de Peralta Menendez, M. M. Murray and S.L. Gonzalez Andino

Abstract— **This paper proposes a new strategy for improving the localization capabilities of linear inverse solutions, based on the relationship between the real solution and the estimated solution as described by the resolution matrix equation. Specifically, we present two alternatives based on either the partial or total inversion of the resolution matrix and applied them to the Minimum Norm solution, which is known for its poor performance in 3D localization problems. The Minimum Norm transformed inverse showed a clear improvement in 3D localization. The strong dependence of localization errors with the eccentricity of the sources, characteristic of this solution, disappears after the proposed transformation. A similar effect is illustrated, using a realistic example where multiple generators at striate areas are active. While the original Minimum Norm incorrectly places the generators at extrastriate cortex, the transformed Minimum Norm localizes, for the example considered, the sources at their correct eccentricity with very low spatial blurring.**

Index Terms—**Inverse Problem, Minimum Norm Solution, Resolution Matrix, Source Localization.**

I. INTRODUCTION

THE neurolectromagnetic inverse problem (NIP), i.e., the reconstruction of the current density vector inside the brain responsible for the electric and magnetic fields measured near/over the scalp, can be represented by a (first kind) Fredholm linear integral equation, denoting the relationship between the data measured at the external point, *d(s)*, and the superposition of the contribution of the unknown current source density distribution at locations *r* inside the brain [1], [2].

$$d(s) = \int_{Brain} L(s,r) \bullet j(r)dr \qquad (1)$$

The (vector) lead field function *L(s,r)* contains all the information about the boundary conditions, as well as the media conductivities or permitivities for the electric and magnetic cases, respectively.

Under experimental conditions, neither the measurements nor the lead field function are known for arbitrary surface/brain locations. However, assuming that the integral equation can be approximated by a discrete sum, (1) can be represented by an underdetermined system of linear equations:

$$\mathbf{d} = \mathbf{L}\mathbf{j} \qquad (2)$$

Vectors **d** and **j** and matrix **L** represent the discretization of the continuous functions, i.e., $\mathbf{d_k} = d(s_k)$, $\mathbf{j_m} = j(r_m)$, and $L_{km} = w_{km}L(s_k, r_m)$ and w_{km} are the quadrature weights. All linear solutions of (2) can be obtained solving a variational problem [3]. This yields the inverse matrix **G** that, when applied to the measured data, produces the estimated current density vector $\hat{\mathbf{j}}$, i.e.:

$$\hat{\mathbf{j}} = \mathbf{G}\mathbf{d} \qquad (3)$$

Substitution of the measured data, as described in (2), into (3) yields the following fundamental equation for underdetermined linear systems:

$$\hat{\mathbf{j}} = \mathbf{G}\mathbf{d} = \mathbf{G}\mathbf{L}\mathbf{j} = \mathbf{R}\mathbf{j} \qquad (4)$$

Here, **R**=**GL** denotes the resolution matrix describing the relationship between the estimates and the original magnitudes. In simpler terms, (4) tells us that our estimates separate from the original values by a factor of **R**. The nearer this factor is to the identity matrix, the better the estimated solution resembles the original sources.

This relationship was initially noted by the Italian mathematician, Giuseppe Peano, in his work related to one particular case of linear functionals, i.e., integral of functions. In 1967, Backus and Gilbert [4] used the same idea to construct solutions to geophysical problems. Since then,

additional applications have appeared that can be included in this framework (e.g. Beamformers, methods [5]) or that generalize this idea to the case of vector fields (WROP as in [6]).

In this paper, we reconsider this relationship for the development of strategies to improve the performance of linear inverse solutions. The idea is very simple. If there is a link between the real sources and the estimates, why not try to improve the estimates to make it closer to the real ones? The following sections detail our proposal.

II. BASIC EQUATIONS

Formally, (4) can be inverted to obtain the original magnitudes from the estimated ones. However-, the rank of the resolution matrix cannot exceed the rank of the lead field matrix **L**. For this reason, and exactly in the same way we do with (2), we have to consider approximate (or generalized) inverses for **R**.

For the particular case we are interested in (i.e. the NIP), different alternatives exist, depending on the interpretation we assign to the blocks of the resolution matrix. Bearing in mind that the unknown vector **j** corresponds to the discretization of a continuous vector field, each group of 3 components is associated with one location in the brain. Consequently, the resolution matrix inherits a particular structure that might influence the inversion strategy selected. In the following, we will present two approaches corresponding to partial and total inversion of the resolution matrix.

A. Partial Inversion Approaches

Consider the construction of an inverse solution aimed at the correct localization of single sources. The first approach consists of approximating the resolution matrix by the 3x3 diagonal blocks. Assuming that the real source corresponds to one single source at location k, only the three components (k-1)*3+1, (k-1)*3+2 and (k-1)*3+3 of vector **j** would differ from zero. Thus, a partial inverse can be obtained by inverting the 3x3 diagonal block associated with point k.

Denote by $\mathbf{R}_{3k}$ the 3x3 diagonal block associated to solution point k and compute $\mathbf{M} = \mathbf{R}_{3k}^{+}$ where the superscript + denotes Moore-Penrose pseudo inversion. The inverse matrix is updated in the following way $\mathbf{G}_k := \mathbf{M}\mathbf{G}_k$, where $\mathbf{G}_k$ stands for the three rows of the inverse associated to point k.

The row-by-row inversion, corresponding to solving the ith equation for the ith component of **j** can be readily calculated. The resulting matricial equation to compute this update of the inverse is $\mathbf{G} := \mathbf{D}[\mathbf{I} - \mathbf{O}]\mathbf{G}$; where **D** is a diagonal matrix composed by the inverse of the elements at the main diagonal of **R** and **O** is a matrix with zeros in the main diagonal and the off-diagonal elements as the resolution matrix.

However, there are many arguments against the use of partial inversion by rows or by blocks. The main drawback is that they are associated with the retrieval of single sources, which we know already does not determine the behavior of a linear inverse for arbitrary source configurations ([7], [8]). In practice, no partial inversion procedure seems to work for all source configurations, for this reason we prefer the strategy described in next section.

B. Total Inversion of the Resolution Matrix

As is usual in numerical mathematics, an approximate inverse matrix can be obtained by regularizing it, i.e., making invertible the matrix by adding a relatively small perturbation. The further away the matrix is from the space of the regular matrices the bigger the perturbation needed to regularize it. Another aspect to consider is the size of the matrix that could obstruct the inversion procedure. A solution that seems to satisfy both previous aspects is to add a diagonal perturbation matrix to the resolution matrix, i.e., invert **(D+R)** where **D** is a perturbation diagonal matrix using the following identity:

$$\mathbf{M} = (\mathbf{D} + \mathbf{R})^{-1} = (\mathbf{D} + \mathbf{GL})^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{G}[\mathbf{I} + \mathbf{L}\mathbf{D}^{-1}\mathbf{G}]^{-1}\mathbf{L}\mathbf{D}^{-1}$$

(5)

This equation considers only the inversion of the diagonal matrix and a matrix of size equal to the number of sensors. Using the previous formula, the update of the inverse matrix **G:= MG** can be performed without the explicit computation of the resolution matrix or its inverse.

III. SIMULATION RESULTS

For reproducibility and compatibility with previous publications, we used a lead field model corresponding to the sensor configuration and solution space described in ISBET NEWSLETTER #6, December 1995, [7], [8]. Specifically, this entailed a unit radius 3-shell spherical head model [9] with solution points confined to a maximum radius of 0.8. The sensor configuration was comprised of 148 electrodes, and the solution space consisted of 817 points on a regular grid with an inter-grid distance of 0.133 cm, corresponding to 2451 focal sources.

The basic figure of merit used to evaluate localization accuracy was the dipole localization error, defined as the Euclidean distance between the point where the maximum of the modulus of the vector field is observed and the actual source position. The errors were divided by the size of the grid unit (0.133) and were evaluated for x values in the set [0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5, 7].

For each value x_i, we computed the empirical Density function and the empirical Probability function defined as:

Density Function(x_i)={ % of sources with errors $\geq x_i$ and $< x_{i+1}$ }*100/2451

Probability Function (x_i)={ % of sources with errors $\leq x_i$}*100/2451

To evaluate possible dependencies of the solution on the depth of the source, we used a measure defined as the average of the dipole localization error for the sources in an eccentricity range. This was the average dipole localization error for sources with eccentricities bigger than or equal to the lower limit and strictly lower than the upper limit. The eccentricity ranges considered were [0.0-0.2), [0.2-0.3), [0.3-0.4), [0.4-0.5), [0.6-0.7), and [0.7-0.8].

To illustrate the improvement that can be obtained with the total inversion procedure, we considered the Minimum Norm solution. This solution corresponds to the Moore Penrose pseudo-inverse and is well known for its poor performance in the localization of single sources. The perturbation matrix, i.e. the diagonal matrix that is added to regularize the resolution matrix was defined as 10 times the absolute value at the main diagonal of the resolution matrix. This resulted in a diagonal-dominant matrix that was obviously invertible.
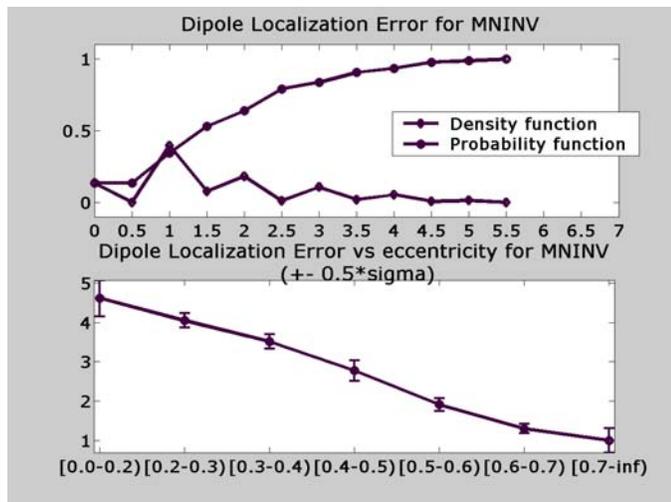

Fig. 1. Results for the Minimum Norm Inverse (MNINV) solution. Upper inset: Empirical distribution function and empirical probability function for the localization error. Lower inset: Average localization error as a function of the eccentricity.

The results for the original Minimum Norm Inverse solution (MNINV) and the Transformed Minimum Norm Inverse (TMNINV) obtained by regularizing the resolution matrix are presented in Fig. 1 and 2, respectively. Readers interested in comparing these localization results with those obtained for additional linear inverses, namely, the minimum Laplacian and the weighted minimum norm solutions are referred to [8]
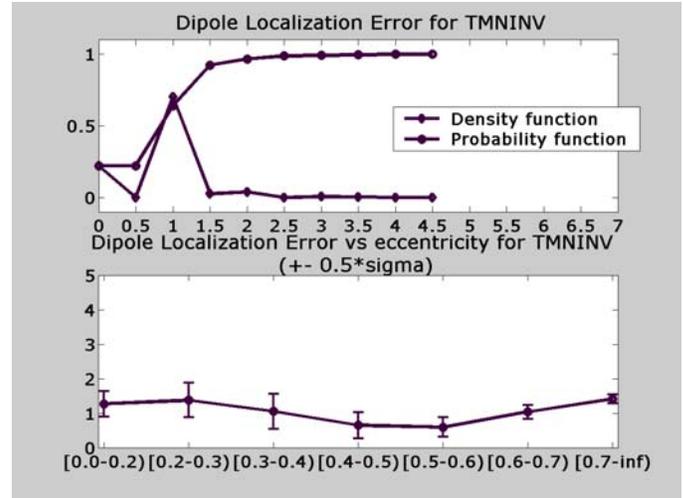
and [10].

Fig. 2. Results for the Transformed Minimum Norm Inverse (TMNINV) solution. Upper inset: Empirical distribution function and empirical probability function for the localization error. Lower inset: Average localization error as a function of the eccentricity.

The transformed inverse (TMNINV) matrix drastically improved its localization capability of single sources with respect to the original inverse (MNINV) as revealed by the following facts:

1. The localization errors of the TMNINV were concentrated in the lower ranges as shown by the number of sources with zero dipole localization that increased from 13% to 22% and the sources with 1 unit error that increased from 34% to 70%.
2. The TMNINV localized more than 92% of the sources with errors lower than 1.5 grid units.
3. The average localization error of the TMNINV was not dependent on the eccentricity and was reduced for all eccentricity values to less than 1.4 grid units.
4. For the TMNINV, the higher localization errors (>2.5) only appeared for less than 1.2% of the sources and the upper limit of maximum localization error decreased from 5.5 to 4.5.

In order to illustrate the practical consequences of using the transformed inverse (TMNINV) instead of the original minimum norm inverse (MNINV) for the localization of multiple simultaneously active sources; we have designed the simple, but nonetheless neurophysiologically plausible, example presented in Fig. 3. In this computer simulation, we calculated both inverses for a solution space of 4024 nodes homogeneously distributed within the inner compartment of a realistic head model derived from the Montreal Neurological Institute average brain (Human Brain Mapping Consortium). The solution space was restricted to the gray matter of this inner compartment and formed a regular grid of six millimeters resolution. The sensor space consisted of 111

electrodes with a spatial distribution resembling the 125-channel Geodesics Sensor Net (Electrical Geodesics Incorporated, Eugene, Oregon) without its lowest electrode line.

Fig. 3a depicts the positions of the three simultaneously active dipoles considered. A dipole was placed within each the left and right lingual gyrus (Brodman area 17) to mimic bilateral activation of primary visual areas (V1), and a third dipole was placed within the left superior frontal gyrus (Brodmann area 10). Note that the frontal dipole is higher along the axial direction than the visual dipoles, which is somewhat obscured by the 3D projection used. All three dipoles had identical dipole moments in the three Cartesian components, which were set to one. In the picture, voxels represented as black circles are the ones where actual or estimated strength exceeds the 85% of the maximum strength.

Fig. 3b shows the localization results obtained for the original minimum norm solution, and Fig. 3c those obtained for the transformed minimum norm. Note that as predictable from its well-known limitations, the original minimum norm solution produces maxima for all the three sources at the borders of the solution space (near the sensors). This will lead to the absolutely erroneous conclusions that activity is elicited at extra striate visual areas. In contrast, the transformed solution correctly indicates bilateral activation of primary visual cortex (striate cortex), although slightly more extended than in the original source distribution, and a third source perfectly localized at the left superior frontal gyrus. Note also that the level of spatial blurring (the number of active nodes at the selected threshold) of the transformed minimum norm solution is considerably smaller than the one present in the original minimum norm reconstruction.
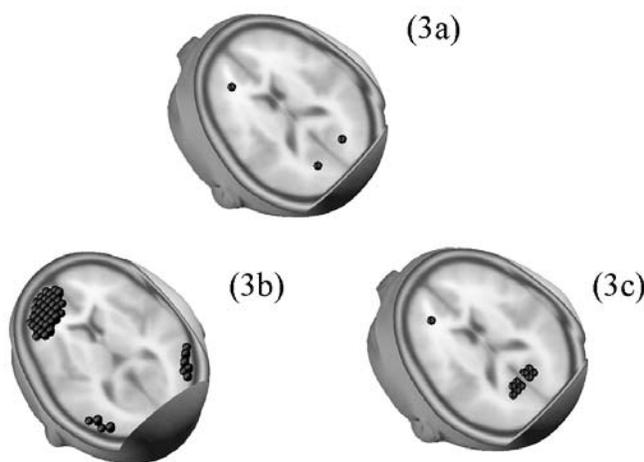

Fig. 3. Localization results for three simultaneously active sources. 3a) The original three dipoles situated at the left superior frontal gyrus and left and right primary visual cortex. Note that the frontal dipole is higher along the axial direction than the visual dipoles, which is somewhat obscured by the 3D projection used. 3b) Localization results for the original minimum norm inverse (MNINV). 3c) Localization results for the transformed minimum norm solution (TMNINV). Note how the original minimum norm incorrectly attribute striate cortex activation to extrastriate visual areas.

Importantly, this example is only illustrative and does not imply that all possible combinations of simultaneously active sources will be localized with the same accuracy.

IV. DISCUSSION

One important point about both the total or partial inversion of the resolution matrix deserves further discussion. Specifically, these procedures almost always transform the inverse matrix in a quasi-inverse manner, in the sense that the original system of equations (2) is no longer solved in the identity. Rather than being disadvantageous, this feature – common to all the Backus and Gilbert type of solutions (e.g. Beamformers and WROP) – seems to be critical for the success of the inverse solution. This separation from the data results in a robust solution that might produce reliable results in the presence of noise. In fact, our experience in the analysis of real data indicates that the updated inverse matrices obtained by partial inversion of (4) are more robust than inverse matrices obtained by regularization of problem (2). One could thus consider this new approximate inverse as a new type of regularized solution of problem (2). In fact, regularization of problem (2) is just another way to separate your inverse from the original system of equations.

This link with regularization methods suggests a clear alternative to select the perturbation matrix in the presence of noise. Representing this perturbation as the product of a diagonal matrix and an unknown factor, i.e. $\mathbf{D} = \alpha\mathbf{W}$, any of the standard methods suggested to compute the regularization parameter α (e.g., [11]) could be applied. The diagonal matrix $\mathbf{W}$ can be set to the identity or to a different value according to some available a priori information, e.g., scaled version of the diagonal of the resolution matrix.

While the idea of inverting the resolution matrix seems to be new, this communication is by no means an exhaustive demonstration of the different inversion strategies that can be used. However, the following theoretical case is worth mentioning. When $\mathbf{G}$ is the Moore Penrose inverse of $\mathbf{L}$ (as the example developed in the previous section), the Moore Penrose inversion of $\mathbf{R} = \mathbf{GL}$ will produce the same inverse matrix and therefore no update. In other words, there are cases where an approximate inverse can perform better than a theoretical one.

To illustrate the method, we considered here the application to the linear neurolectromagnetic inverse problem. However, the procedures proposed here are equally valid for any inverse problem with a resolution operator associated to it.

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Rolando Grave de Peralta Menendez Menendez was born in Santiago de Cuba, Cuba on November 30, 1962. He received the M. Sc. degree in Mechanical Engineering (with honors) from the Technical University of Havana (ISPJAE) in 1987, the M. Sc. degree in Mathematics from the University of Havana in 1992, and the PhD degree in Sciences from Geneva University in 1998.

He was at the National Center for Scientific Research, Cuba, from 1987 to 1995, when moved to the Institute of Experimental Audiology, University of Münster, Germany. Since 1996, he has been working as a specialist on the inverse problem, at the Functional Brain Mapping Lab, University Hospital of Geneva, Switzerland, leading the research on inverse problem and its applications. His research interests center on Applied Mathematics namely: Numerical Analysis, Global Optimization Techniques and Inverse Problems with discrete data and its applications in biomedicine. These applications concern the development of a biophysically constrained brain electromagnetic tomography for source localization, direct non-invasive brain computer interfaces and stroke and epilepsy monitoring systems.

Micah M. Murray was born 23 November 1973 in Danvers, Massachusetts. In 1995, he received a B.A. from the Johns Hopkins University in both psychology and English. In 1999, he received a M.S. in neuroscience and in 2001 a Ph.D. in neuroscience with honors from the Albert Einstein College of Medicine of Yeshiva University. His thesis work was conducted within the Cognitive Neurophysiology Laboratory of the Nathan S. Kline Institute for Psychiatric Research in New York.

After his Ph.D. he worked as a post-doctoral scientist in the Functional Brain Mapping Laboratory of the Neurology Clinic as well as within the Department of Rehabilitation, both of which are at the University Hospital of Geneva. Since August of 2003, he has been the head of the Functional Electrical Neuroimaging Laboratory within the Neuropsychology Division and Radiology Service of the Centre Hospitalier Universitaire Vaudois in Lausanne, Switzerland. Research interests of the lab include multisensory interactions, electrophysiologic and hemodynamic (EEG-BOLD) signal coupling, parallel processing and plasticity in the auditory system, and object recognition. In addition to fMRI, the lab uses EEG as a neuroimaging technique to spatio-temporally localize sensory, perceptual, and cognitive functions in the healthy and pathological brain.

Sara L. Gonzalez Andino was born in Havana, Cuba on November 8, 1963. She received the M.Sc. degree in Physics (with honors) from the University of Havana in 1985 and the PhD degree in Sciences from the University of Geneva in 2001.

She was an Associated Researcher and Assistant Professor at the Cuban National Center for Scientific Research from 1985 to 1995. From 1989 to 1990 she was working as a visitor researcher at the Institute of Experimental Audiology, University of Münster, Germany. Since 1996, she has been working at the Functional Brain Mapping Lab, University Hospital of Geneva, Switzerland. Her research interests include biophysical modeling, inverse and direct problem solution in biomedicine, Brain Computer Interface, and studies about coherence and synchrony in the human brain.